

EE 3302-502, Course: Signals and Systems
Class Lecture Notes

Textbooks: Signals & Systems
 by
 A.V. Oppenheim, A. S. Willsky, S. H. Nawab
 2nd Edition, Prentice Hall

 and
Continuous and Discrete Signals and Systems
 by
 S. S. Soliman, M. D. Srinath
 2nd Edition, Prentice Hall

Instructor: Dr. Issa Panahi

NOTE:

**Notations and symbols used in these lecture notes
are the same as those in Soliman's book.
These notations are slightly different from those in
Oppenheim's book.**

Representing Signals

- * Signals are information carrying quantities that are detectable, measurable or are estimated.
- * A signal is function of single independent variable - called one dimensional, or multiple independent variables - called multi-dimensional signal.
- * A signal can be function of another signal - called single channel, or multiple signals - called multi-channel signal.
- * A signal can be deterministic or random.
- * Deterministic signal is the one that knowing something about it, like its past, enable us to know everything about it, like its future values, EXACTLY.
- * Random signal is the one that knowing something about it, like its past, can not enable us to know everything about it, like its future values EXACTLY. Some random signals may allow us to just predict or estimate their behavior or values in some ways.

* Examples:

$$(1) x(t) = A \cos(2\pi f_0 t + \phi) = A \cos(\omega_0 t + \phi)$$

t : Time in second

f_0 : Frequency in Hertz (Hz)

$\omega_0 = 2\pi f_0$: Frequency in Radian per second

ϕ : phase in Radian

A : Amplitude (in some quantity, like voltage)

$$(2) y(t) = A e^{-\alpha t}$$

$$(3) u(t) = t^2 + \alpha t + \beta$$

$$(4) v(t, d) = \sigma t - 2d + A \cos(\omega_0 t + \phi)$$

$$(5) r(t, d) = \begin{bmatrix} x(t) \\ v(t, d) \\ y(t) \end{bmatrix}$$

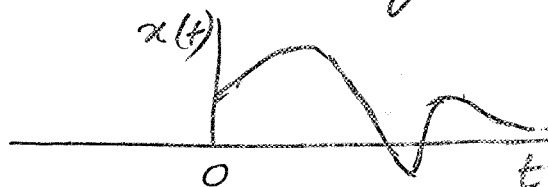
Above signals are deterministic, single variable, multi-variable, single channel, and multi-channel signals.

(6) $w(t)$: white noise. is single variable random signal.

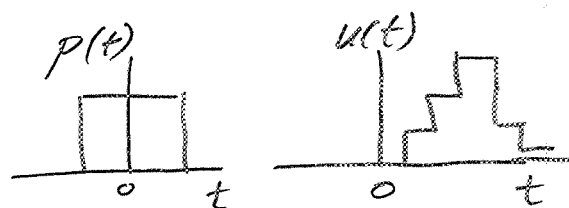
* In this Course we only deal with deterministic signals, as function of single independent variable time or frequency.

Other signal classifications

* Continuous-time signals:

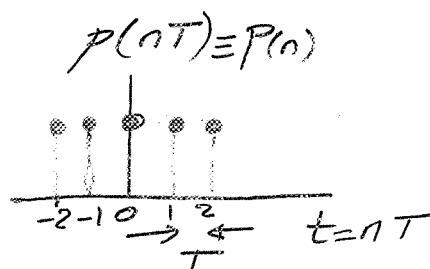
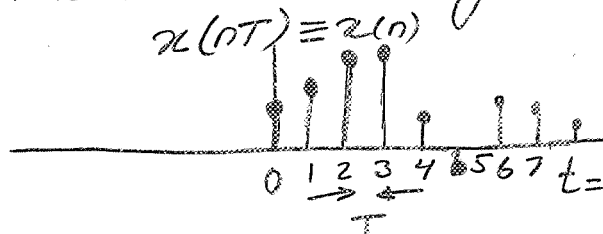


Continuous signal



Piece wise Continuous-time signals

* Discrete-time signals:



$x(nT) = x(t)$ for $t = nT$, T is sampling period.

Ex $y(t) = A \sin(\omega_0 t + \theta)$; CT

$$y(t) \Big|_{t=nT} = y_s(t) = y(nT) = A \sin(\omega_0 nT + \theta), \quad \text{DT}$$

* Periodic signal $x(t)$: $x(t)$ is periodic iff $x(t) = x(t + nT)$, T is fundamental period.

$$n = +/ - 1, 2, 3, 4, \dots$$

Ex 1 $x(t) = A \cos(\omega_0 t + \phi)$
 $= A \cos(2\pi f_0 t + \phi)$ since $\omega_0 = 2\pi f_0$

To see if $x(t)$ is periodic, and to find T we must check the condition $x(t) = x(t+T)$:

$$\begin{aligned} x(t+T) &= A \cos(2\pi f_0 (t+T) + \phi) \\ &= A \cos[2\pi f_0 t + \phi + 2\pi f_0 T] \end{aligned}$$

Using

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

we get

$$x(t+T) = A \left\{ \cos(2\pi f_0 t + \phi) \cos(2\pi f_0 T) - \sin(2\pi f_0 t + \phi) \sin(2\pi f_0 T) \right\}$$

Now for $T = \frac{1}{f_0}$ we have:

$$\cos(2\pi) = 1, \quad \sin(2\pi) = 0, \quad \text{Thus}$$

$$x(t+T) = A \cos(2\pi f_0 t + \phi) = x(t); \text{ Periodic}$$

and $T = \frac{1}{f_0}$ is the fundamental period.

Ex. 2

$$y(t) = A \sin(2\pi f_1 t) - B e^{-j 2\pi f_2 t}$$

Note that:

$$e^{\mp j\theta} = \cos \theta \mp j \sin \theta$$

Thus

$$y(t) = A \sin(2\pi f_1 t) - B \cos(2\pi f_2 t) + B \sin(2\pi f_2 t)$$

The fundamental period for $y_1(t) = \sin(2\pi f_1 t)$

$$\text{is } T_1 = \frac{1}{f_1}$$

For $y_2(t) = \cos(2\pi f_2 t)$ we have $T_2 = \frac{1}{f_2}$ and for $y_3(t) = \sin(2\pi f_2 t)$ we have

$$T_3 = \frac{1}{f_2} = T_2$$

$$\text{Thus } y_2(t + nT_2) = y_2(t), \quad y_3(t + nT_2) = y_3(t)$$

$$\text{and } y_1(t + kT_1) = y_1(t).$$

Now we must find T such that

$$y(t+T) = y(t) = A y_1(t + kT_1) - B y_2(t + nT_2) + B y_3(t + nT_2)$$

$$\text{or } T = kT_1 = nT_2$$

$$\text{Thus } \frac{T_1}{T_2} = \frac{n}{k}$$

a rational number since n and k must be integers.therefore $y(t)$ is periodic if $\frac{T_1}{T_2}$ is a rational number.

Thus we must find $\frac{T_1}{T_2}$, simplify it, and see if it yields a rational number. If it does then n and k , and therefore T is found.

Ex 3 $u(t) = \cos(\omega_0 t) \sin(\omega_1 t + \theta)$

Then use trigonometric relation to write this as a sum of terms;

$$\cos(\alpha) \sin(\beta) = \frac{1}{2} \sin(\alpha + \beta) - \frac{1}{2} \sin(\alpha - \beta)$$

$$u(t) = \frac{1}{2} \sin[(\omega_0 + \omega_1)t + \theta] - \frac{1}{2} \sin[(\omega_0 - \omega_1)t - \theta]$$

Now we need to do the same as in Ex 2 for two sines in the right hand side.

Note if $\omega_0 - \omega_1 = 0$ then

$$\sin[(\omega_0 - \omega_1)t - \theta] = -\sin(\theta) = A \text{ which}$$

is a constant (i.e not function of t).

and hence

$$u(t) = \frac{1}{2} \sin(2\omega_0 t + \theta) + \frac{A}{2}$$

where $T = \frac{1}{2f_0}$, $\omega_0 = 2\pi f_0$

and $u(t) = u(t+T)$ is periodic.

Energy and power of a signal

For a real valued signal $x(t)$,

* $x^2(t)$ is instantaneous power.

* $E_{2L} = \int_{-L}^L |x(t)|^2 dt$ is signal energy over a time interval of length $2L$.

* $E = \lim_{L \rightarrow \infty} \int_{-L}^L |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$
is total Energy of $x(t)$.

* $P = \lim_{L \rightarrow \infty} \left[\frac{1}{2L} \int_{-L}^L |x(t)|^2 dt \right]$ is total average power of $x(t)$.

Now
(1) A signal, $x(t)$, is called Energy signal if E exists, i.e., E has a finite value.
($0 < E < \infty$).

(2) If E exists for $x(t)$ then average power of $x(t)$ is zero.

(3) If energy E of $x(t)$ is infinite, i.e. it does not exist, then $x(t)$ may or may not have average power P .

(4) If $E = \infty$ but $x(t)$ has finite average power P (i.e. $0 < P < \infty$) then $x(t)$ is called power signal.

(5) A periodic signal $x(t) = x(t+T)$ is a power signal, i.e. its energy is infinite but its average power is finite given by $P = \frac{1}{T} \int_0^T |x(t)|^2 dt$.

Ex 1: $x(t) = A \cos(\omega_0 t + \theta)$, $T = \frac{1}{f_0}$
has infinite Energy, but $\omega_0 = 2\pi f_0$

$$P = \frac{1}{T} \int_0^T A^2 \cos^2(\omega_0 t + \theta) dt$$

$$P = \frac{A^2}{T} \int_0^T \left[\frac{1}{2} + \frac{1}{2} \cos(2\omega_0 t + 2\theta) \right] dt$$

$$= \frac{A^2}{2} + \frac{A^2}{2T} \int_0^T \cos(2\omega_0 t + 2\theta) dt$$

$$= \frac{A^2}{2} + \frac{A^2}{2T} \left[\frac{1}{2\omega_0} \sin(2\omega_0 t + 2\theta) \right]_0^T$$

$$= \frac{A^2}{2} + \frac{A^2}{2T} \left(\frac{1}{2\omega_0} \right) \left\{ \underbrace{\sin(4\pi + 2\theta)}_0 - \sin(2\theta) \right\}$$

$$P = \frac{A^2}{2}$$

Ex 2: $x(t) = B e^{-\alpha t}$ for $t \geq 0$

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} B^2 e^{-2\alpha t} dt \\ &= B^2 \int_0^{\infty} e^{-2\alpha t} dt \\ &= B^2 \left(\frac{1}{-2\alpha} \right) \left[e^{-2\alpha t} \right]_{t=0}^{\infty} \end{aligned}$$

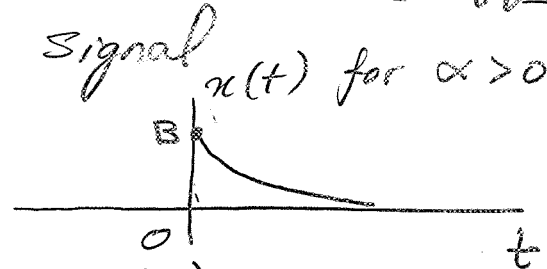
$$E = \frac{B^2}{-2\alpha} \left\{ e^{-2\alpha(\infty)} - 1 \right\}$$

Now

(1)

if $\alpha > 0$ then $e^{-2\alpha(\infty)} = 0$ and $x(t)$ is Energy signal

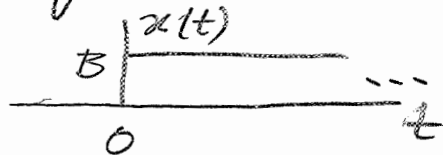
$$E = \frac{B^2}{2\alpha}$$



(2) If $\alpha < 0$ then $e^{-2\alpha(\infty)} = \infty$ and $x(t)$ is of infinite energy, that is $x(t)$ is not Energy signal

(3) If $\alpha \neq 0$, i.e. positive or negative value of α , then $x(t)$ is not a power signal since its Average power is zero.

(4) If $\alpha = 0$ then $x(t) = B$ for $t \geq 0$;



$$\text{Then } E = \int_0^{\infty} |x(t)|^2 dt = B^2 \int_0^{\infty} dt$$

$$= B^2 [\infty - 0] = \infty \Rightarrow \text{Not Energy Signal.}$$

However the Average power is:

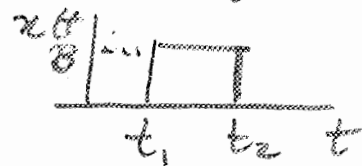
$$P = \lim_{L \rightarrow \infty} \left[\frac{1}{2L} \int_0^L B^2 dt \right]$$

$$= \lim_{L \rightarrow \infty} \left[\frac{B^2}{2L} (L - 0) \right] = \lim_{L \rightarrow \infty} \frac{B^2}{2}$$

$$P = \frac{B^2}{2} < \infty$$

Thus $x(t) = B$ for $t \geq 0$ is a Power signal, but it is NOT periodic.

Ex 3: $x(t) = B$ for $t_1 \leq t \leq t_2$ and zero elsewhere.



Then $x(t)$ is not periodic.

$$\text{But } E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{t_1}^{t_2} B^2 dt = B^2 (t_2 - t_1)$$

and $x(t)$ is Energy signal.

Thus, any time limited signal is Energy signal.
(bounded)

Transformations of the Independent Variable

* Shifting operation:

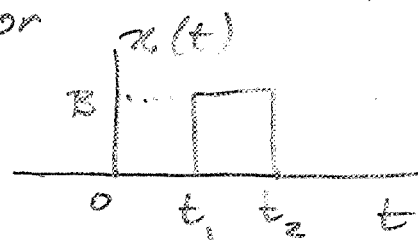
Given $x(t)$ then

(1) $y(t) = x(t - \alpha)$ is shift to the right of
for $\alpha > 0$. or delaying $x(t)$ by α .

(2) $v(t) = x(t + \alpha)$ is shift to the left
for $\alpha > 0$. of or advancing
 $x(t)$ by α .

Ex $x(t) = B$ for $t_1 \leq t \leq t_2$ or

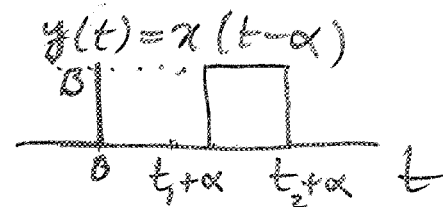
$$x(t) = \begin{cases} B & t_1 \leq t \leq t_2 \\ 0 & \text{elsewhere} \end{cases}$$



Now

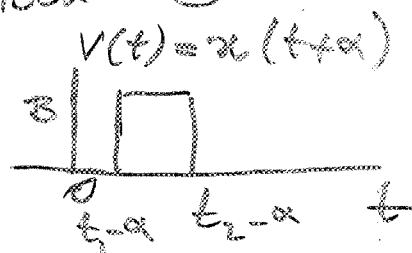
* $y(t) = x(t - \alpha)$, for $\alpha > 0$:

$$y(t) = \begin{cases} B & t_1 \leq t - \alpha \leq t_2 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} B & t_1 + \alpha \leq t \leq t_2 + \alpha \\ 0 & \text{elsewhere} \end{cases}$$



* $v(t) = x(t + \alpha)$, for $\alpha > 0$:

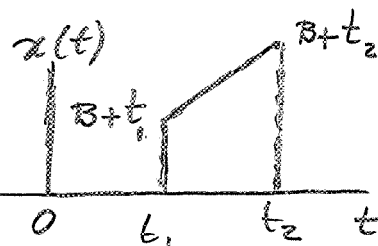
$$v(t) = \begin{cases} B & t_1 \leq t + \alpha \leq t_2 \\ 0 & \text{elsewhere} \end{cases} = \begin{cases} B & t_1 - \alpha \leq t \leq t_2 - \alpha \\ 0 & \text{elsewhere} \end{cases}$$



* Time Reflection operation (or Folding):

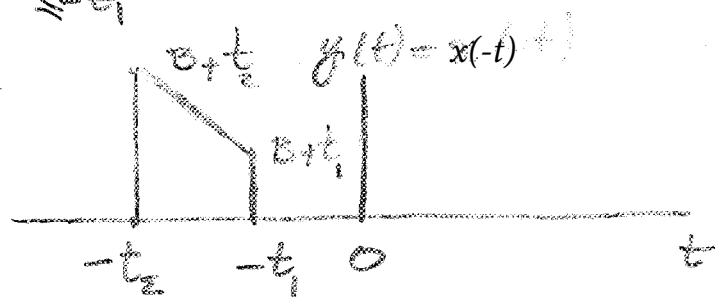
Given $x(t)$ then $y(t) = x(-t)$ is the reflected or folded version of $x(t)$ around $t=0$.

Ex. $x(t) = \begin{cases} B+t & \text{for } t_1 \leq t \leq t_2 \\ 0 & \text{elsewhere} \end{cases}$



$$y(t) = x(-t) = \begin{cases} B-t & t_1 \leq -t \leq t_2 \\ 0 & \text{elsewhere} \end{cases}$$

or $y(t) = \begin{cases} B-t & -t_2 \leq t \leq -t_1 \\ 0 & \text{elsewhere} \end{cases}$



* Time-Scaling operation:

Given $x(t)$ then $y(t) = x(\alpha t)$ is the time-scaled version of $x(t)$ by factor α .

Ex 1: $x(t) = \sin(2\pi f_0 t) = \sin\left(\frac{2\pi}{8} t\right)$, $t \geq 0$

where $f_0 = \frac{1}{T} = \frac{1}{8}$, T is the period,
 $T = 8$.

then $y(t) = x(\alpha t)$.

* Now let $\alpha = 2$ then $y(t) = \sin\left(\frac{2\pi}{8} 2t\right)$

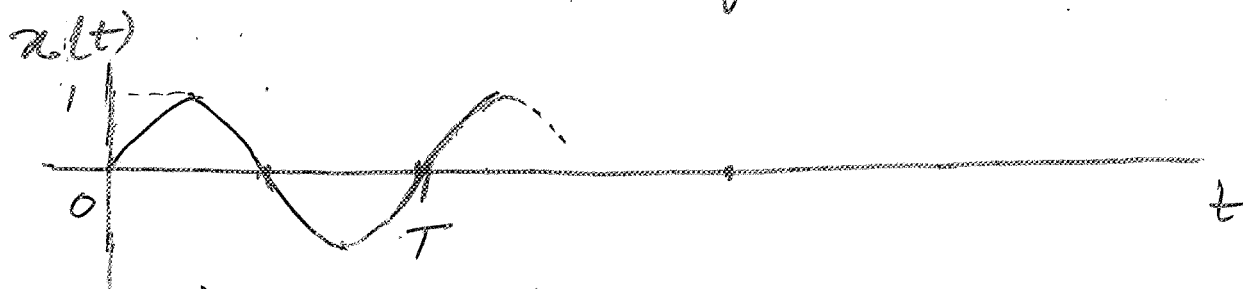
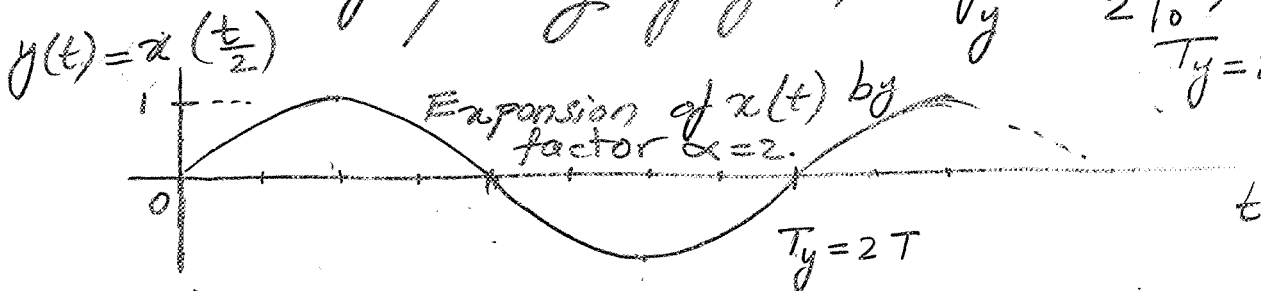
and $y(t) = x(2t) = \sin\left(\frac{2\pi}{4} t\right)$,

Thus the frequency of $y(t)$ is $f_y = \frac{1}{4} = \alpha f_0$.
and its period is $T_y = 4 = \frac{T}{2}$

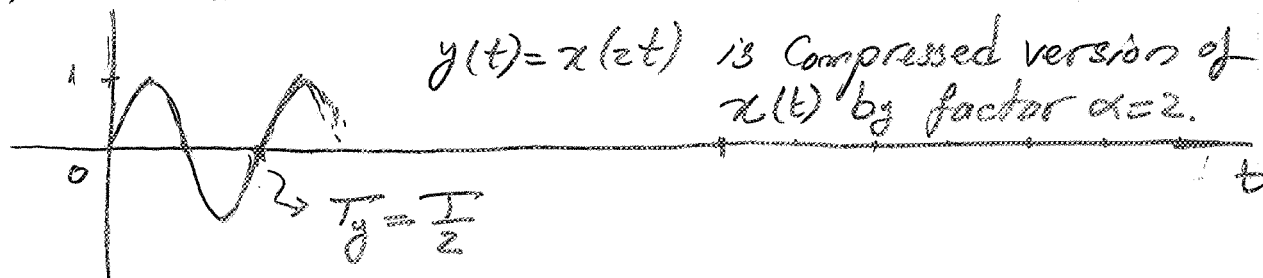
* Now let $\alpha = \frac{1}{2}$ then $y(t) = \sin\left(\frac{2\pi}{8} \frac{1}{2} t\right)$

and $y(t) = x\left(\frac{t}{2}\right) = \sin\left(\frac{2\pi}{16} t\right)$,

Thus the frequency of $y(t)$ is $f_y = \frac{1}{2} f_0$,
 $T_y = 2T$



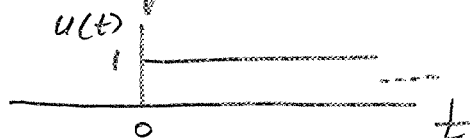
$y(t) = x(2t)$



Elementary Signals

* Good for testing and representing signals and systems.

(1) Unit step function: $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

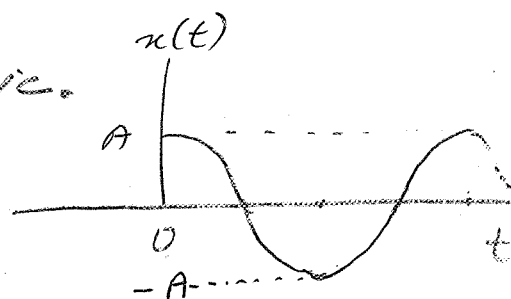


Ex.

$$x(t) = A \cos(\omega_0 t + \theta) u(t) = y(t) \cdot u(t)$$

Where $y(t) = A \cos(\omega_0 t + \theta)$ for $-\infty \leq t \leq \infty$ is a periodic signal.

Note: $x(t)$ is not periodic.

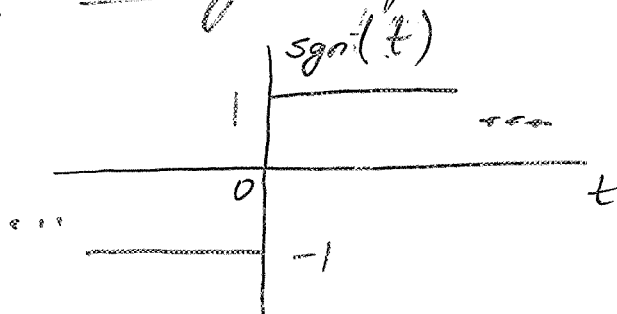


(2) The Ramp function:

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where $r(t) = \int_{-\infty}^t u(\tau) d\tau$

(3) The Signum function:

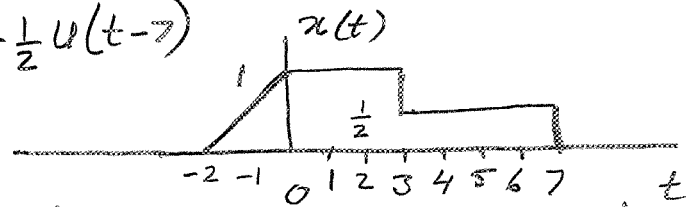


$$\text{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$\text{sgn}(t) = u(t) - u(-t)$$

$$\text{or } = -1 + 2u(t)$$

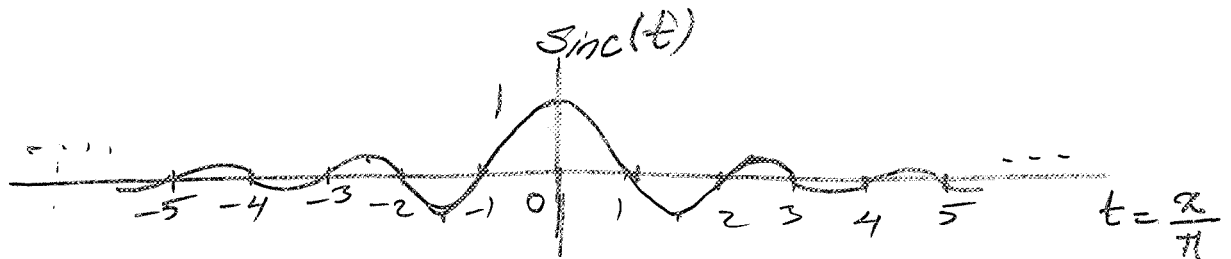
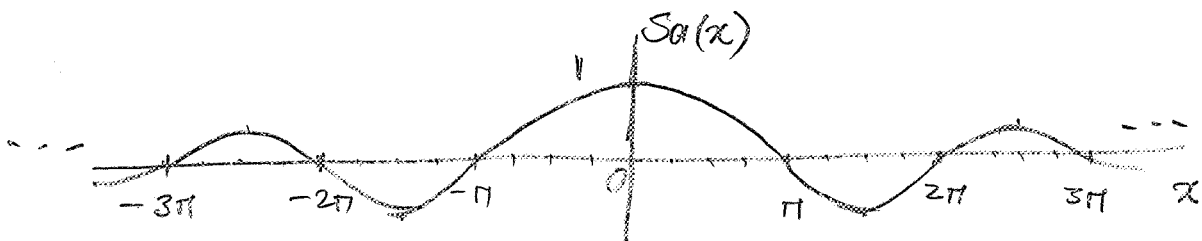
Ex: $x(t) = \frac{1}{2} r(t+2) [u(t+2) - u(t)] + u(t) - \frac{1}{2} u(t-3) - \frac{1}{2} u(t-7)$



(4) The Sampling function and The Sinc function:

$$Sa(x) = \frac{\sin(x)}{x} \quad x \text{ is in radian here.}$$

$$\text{Sinc}(t) = \frac{\sin(\pi t)}{\pi t} = Sa(\pi t)$$



Properties: * $Sa(0) = 1$, $\text{Sinc}(0) = 1$

* Half of main lobe width is the same as the width of each sidelobe.

* Zero-crossing at $x = t\pi$ where $t = \pm 1, \pm 2, \pm 3, \dots$

(5) The Unit Impulse function, $\delta(t)$:

* Also known as Dirac Delta function or Delta function.

$$* \delta(t) = \begin{cases} \infty & t=0 \\ 0 & t \neq 0 \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(t) dt = 1$$

* $\delta(t) = \delta(-t)$ i.e. an even function,

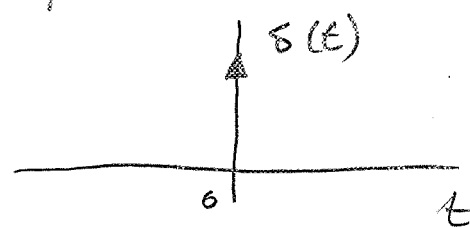
* $\delta(t)$ has finite Energy = 1

$$* \int_{t_1}^{t_2} x(t) \delta(t) dt = x(0) \quad t_1 < 0 < t_2$$

for any function $x(t)$.

$$* \int_{t_1}^{t_2} x(t) \delta(t) dt = 0$$

for t outside t_2 and t_1 interval.



Important properties of $\delta(t)$:

(a) Sifting property:

$$\int_{t_1}^{t_2} x(t) \delta(t-t_0) dt = \begin{cases} x(t_0) & t_1 < t_0 < t_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{or } x(t) = \int_{-\infty}^{\infty} x(t) \delta(t-\tau) d\tau$$

$$\text{Also } \int_{t_1}^{t_2} x(t) \delta(t-t_0) dt = \frac{1}{2} x(t_0) \text{ for } t_0 = t_1 \text{ or } t_0 = t_2$$

(b) Sampling property:

$$x(t) \delta(t-t_0) = x(t_0) \delta(t-t_0)$$

(c) scaling property:

$$\delta(at+b) = \frac{1}{|a|} \delta\left(t + \frac{b}{a}\right)$$

(d) $\int_{-\infty}^t \delta(\tau) d\tau = u(t) \text{ for } t > 0.$

(e) Derivatives of $\delta(t)$:

$$\int_{t_1}^{t_2} x(t) \delta^{(n)}(t-t_0) dt = (-1)^n x^{(n)}(t_0)$$

for $t_1 < t_0 < t_2$

for any $x(t)$ that has finite n th derivative at $t=t_0$.

First derivative of $\delta(t)$ is at $n=1$ or

$$\delta^{(1)}(t) \equiv \delta'(t).$$

